Backlund transformations for the Nizhnik-Novikov-Veselov equation.

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Abstract

The Backlund transformations for the Nizhnik-Novikov-Veselov equation are presented. It is shown that these transformations can be iterated and that the resulting sequence can be described by the Volterra equations. The relationships between the Nizhnik-Novikov-Veselov equation and the Volterra hierarchy are discussed.

1 Introduction.

In this paper I want to discuss the Backlund transformations (BTs) for the asymmetric version of the Nizhnik-Novikov-Veselov equation (NNVE) [1, 2, 3],

$$\left\{ \begin{array}{rcl} 0 & = & -p_t + p_{xxx} + 6\,(wp)_x \\ 0 & = & p_x + w_y. \end{array} \right. \tag{1.1}$$

The BTs which will be discussed below belong to the class of transformations studied in [4, 5, 6, 7]. Contrary to the Backlund-Darboux (or soliton-adding) transformations they do not possess the superposition property but appear in sequences that can be described by some discrete equations, which turn out to be integrable as well as the original system.

I am going to start with the traditional approach: in section 2, I present a set of relations and prove (in appendix A) that they indeed link different solutions of the NNVE. Then by introducing some additional functions I rewrite these relations as a set of bilinear equations which can be shown to belong to the Volterra hierarchy (VH) (section 3). To demonstrate the relations between the NNVE and the VH, I derive the former from the the latter (section 4), i.e. I show that the NNVE can be obtained as a differential consequence of equations of the VH. Finally, I discuss the derived BTs in the framework of the zero curvature reprezentation (section 5).

2 Backlund transformations.

In what follows I use instead of p and w from (1.1) the corresponding tau-function. To introduce it we first 'solve' the second equation of (1.1) by presenting p and w as

$$p = -\lambda_{xy} \qquad w = \lambda_{xx}. \tag{2.1}$$

In terms of λ the system (1.1) becomes

$$\lambda_{ty} - \lambda_{xxxy} - 6\lambda_{xx}\lambda_{xy} = 0 \tag{2.2}$$

(here I have omitted a term which does not depend on x and can be eliminated by a symmetry transform). In terms of the function τ given by

$$\tau = \exp\lambda \tag{2.3}$$

equation (2.2) can be rewritten in the bilinear form

$$\left(D_{ty} - D_{xxxy}\right)\tau\cdot\tau = 0 \tag{2.4}$$

which is the simplest DKP equation, according to the classification of Jimbo and Miwa [8]. Here the symbol D stands for the Hirota's bilinear operators

$$D_x^m D_y^n ...\; a\cdot b = \frac{\partial^m}{\partial\xi^m}\frac{\partial^n}{\partial\eta^n}\; ...\; a(x+\xi, y+\eta, ...)b(x-\xi, y-\eta, ...)\big|_{\xi=\eta=...=0}\,. \tag{2.5}$$

Using the tau-function τ and its logarithm λ the central result of the paper (the BT for the NNVE) can be formulated as follows: if two tau-functions, τ and $\hat{\tau}$, are related by

$$\Lambda_t = \Lambda_{xxx} + \Lambda_x^3 + 3\Lambda_x M_{xx} + \frac{3}{\Lambda_x}\lambda_{xx}\hat{\lambda}_{xx} \tag{2.6}$$

$$0 = \lambda_{xy}\hat{\lambda}_{xy} - \Lambda_x \tag{2.7}$$

$$0 = \lambda_{xxy}\hat{\lambda}_{xy} - \lambda_{xy}\hat{\lambda}_{xxy} + 2\Lambda_x^2 - M_{xx} \tag{2.8}$$

where

$$\Lambda = \lambda - \hat{\lambda} \qquad M = \lambda + \hat{\lambda} \tag{2.9}$$

with

$$\lambda = \ln\tau \qquad \hat{\lambda} = \ln\hat{\tau} \tag{2.10}$$

and one of them, say τ, solves (2.4), then the other one, $\hat{\tau}$, is also a solution of (2.4).

Of course, expressions (2.6)–(2.8) are rather cumbersome and, if one wants to prove that (i) this system of three equations for two functions is compatible or that (ii) these relations are indeed BTs, one needs rather lengthy (though not very difficult) calculations. In appendix A I give a direct proof of the statement (ii). As to the question (i) I will return to it in section 4 when our BT will be reformulated in a more elegant and transparent form.

3 From NNVE to VH.

The aim of this section is to show that BT (2.6)–(2.8) can be described by equations from the VH. To do this we need to introduce some quantities which enable us to reformulate the BTs discussed above in a more transparent way. Consider functions σ, $\hat{\sigma}$ and ρ, $\hat{\rho}$ defined by

$$\sigma = -\frac{\tau^2}{\hat{\tau}}\lambda_{xy} \qquad \hat{\sigma} = -\frac{\hat{\tau}^2}{\tau}\hat{\lambda}_{xy} \tag{3.1}$$

and

$$\rho = -\frac{\tau^3}{\hat{\tau}^2}\left(\lambda_{xxy} + \Lambda_x\lambda_{xy}\right) \qquad \hat{\rho} = \frac{\hat{\tau}^3}{\tau^2}\left(\hat{\lambda}_{xxy} - \Lambda_x\hat{\lambda}_{xy}\right). \tag{3.2}$$

Noting that

$$\sigma\hat{\sigma} = \tau\hat{\tau}\ \lambda_{xy}\hat{\lambda}_{xy} = \tau\hat{\tau}\ \left(\ln\frac{\tau}{\hat{\tau}}\right)_x \tag{3.3}$$

(I have used (2.7)) one can rewrite the last equation and the definitions of ρ and $\hat{\rho}$ in a similar bilinear way:

$$\begin{aligned} D_x\,\sigma\cdot\tau &= \rho\hat{\tau} \\ D_x\,\tau\cdot\hat{\tau} &= \sigma\hat{\sigma} \\ D_x\,\hat{\tau}\cdot\hat{\sigma} &= \tau\hat{\rho}. \end{aligned} \tag{3.4}$$

One can also obtain a set of bilinear equations involving y-derivatives. Differentiating (3.1) with respect to y we get

$$D_y\,\sigma\cdot\hat{\tau} - \tau^2 = -\tau^2\,\Theta \tag{3.5}$$

$$D_y\,\hat{\sigma}\cdot\tau + \hat{\tau}^2 = -\hat{\tau}^2\,\hat{\Theta} \tag{3.6}$$

where

$$\Theta = \lambda_{xyy} + 2\Lambda_y\lambda_{xy} + 1 \tag{3.7}$$

$$\hat{\Theta} = \hat{\lambda}_{xyy} - 2\Lambda_y\hat{\lambda}_{xy} - 1 \tag{3.8}$$

while equations (3.2) lead to

$$D_y\,\rho\cdot\tau - \sigma^2 = -\frac{\tau^4}{\hat{\tau}^2}\left(\Theta_x + \Lambda_x\Theta\right) \tag{3.9}$$

$$D_y\,\hat{\rho}\cdot\hat{\tau} + \hat{\sigma}^2 = \frac{\hat{\tau}^4}{\tau^2}\left(\hat{\Theta}_x - \Lambda_x\hat{\Theta}\right). \tag{3.10}$$

To calculate the quantities Θ and $\hat{\Theta}$ one can obtain, by differentiating (2.7) and (2.8) with respect to y, the following identities:

$$\hat{\lambda}_{xy}\Theta + \lambda_{xy}\hat{\Theta} = 0 \tag{3.11}$$

$$\hat{\lambda}_{xy}\Theta_x - \hat{\lambda}_{xxy}\Theta - \lambda_{xy}\hat{\Theta}_x + \lambda_{xxy}\hat{\Theta} = 0 \tag{3.12}$$

which give

$$\left(\frac{\Theta}{\lambda_{xy}}\right)_x = 0 \qquad \text{and} \qquad \left(\frac{\hat{\Theta}}{\hat{\lambda}_{xy}}\right)_x = 0. \tag{3.13}$$

Hence,

$$\Theta = \varphi\lambda_{xy} \qquad \text{and} \qquad \hat{\Theta} = \hat{\varphi}\hat{\lambda}_{xy} \tag{3.14}$$

where φ and $\hat{\varphi}$ are some functions which do not depend on x,

$$\varphi_x = \hat{\varphi}_x = 0. \tag{3.15}$$

Returning to (3.11) one can conclude that $\hat{\varphi} = -\varphi$. This leads, together with (3.1), to the following result for Θ and $\hat{\Theta}$:

$$\Theta = -\varphi\frac{\sigma\hat{\tau}}{\tau^2} \qquad \text{and} \qquad \hat{\Theta} = \varphi\frac{\tau\hat{\sigma}}{\hat{\tau}^2}. \tag{3.16}$$

Finally, one can rewrite (3.9), (3.5), (3.6) and (3.10) as

$$\begin{array}{rcl} D_y\,\rho\cdot\tau - \varphi\,\rho\tau & = & \sigma^2 \\ D_y\,\sigma\cdot\hat{\tau} - \varphi\,\sigma\hat{\tau} & = & \tau^2 \\ D_y\,\tau\cdot\hat{\sigma} - \varphi\,\tau\hat{\sigma} & = & \hat{\tau}^2 \\ D_y\,\hat{\tau}\cdot\hat{\rho} - \varphi\,\hat{\tau}\hat{\rho} & = & \hat{\sigma}^2. \end{array} \tag{3.17}$$

One can clearly see that our six functions form a chain

$$\hat{\rho} \to \hat{\sigma} \to \hat{\tau} \to \tau \to \sigma \to \rho. \tag{3.18}$$

Indeed, after introducing the sequence of tau-functions τ_n,

$$\begin{array}{rclcrcl} \tau_1 & = & \tau & \qquad & \tau_0 & = & \hat{\tau}\exp(\Phi) \\ \tau_2 & = & \sigma\exp(-\Phi) & & \tau_{-1} & = & \hat{\sigma}\exp(2\Phi) \\ \tau_3 & = & \rho\exp(-2\Phi) & & \tau_{-2} & = & \hat{\rho}\exp(3\Phi) \end{array} \tag{3.19}$$

where Φ is an antiderivative of $\varphi/2$,

$$\Phi_y = \frac{\varphi}{2} \tag{3.20}$$

equations (3.4) and (3.17) become

$$D_x\,\tau_{n+1}\cdot\tau_n = \tau_{n+2}\tau_{n-1} \qquad n = -1, 0, 1 \tag{3.21}$$

and

$$D_y\,\tau_{n+1}\cdot\tau_{n-1} = \tau_n^2 \qquad n = -1, 0, 1, 2. \tag{3.22}$$

It can be shown that the sequence $\tau_{-2} \to ... \to \tau_2$ can be extended in both directions to infinity: we can introduce functions τ_n for $n = \pm 3, \pm 4, ...$ in such a way that they will satisfy (3.21) and (3.22),

$$D_x\,\tau_{n+1}\cdot\tau_n = \tau_{n+2}\tau_{n-1} \tag{3.23}$$

and

$$D_y\,\tau_{n+1}\cdot\tau_{n-1} = \tau_n^2 \tag{3.24}$$

for every n (see appendix B).

One can easily recognize in (3.23) the classical Volterra chain written in the bilinear form. Indeed, the quantities u_n defined by

$$u_n = \frac{\tau_{n-2}\tau_{n+1}}{\tau_{n-1}\tau_n} \tag{3.25}$$

solve the famous Volterra equation

$$\partial_x u_n = u_n \left(u_{n+1} - u_{n-1} \right) \tag{3.26}$$

(where $\partial_x = \partial / \partial x$). As to equations (3.24), they are nothing but the simplest equations of the negative Volterra hierarchy discussed in [9]. Finally, the t-equation (2.6) of our BT is, in the terms of the tau-functions τ_n, the *third* equation of the classical (positive) VH,

$$\left(D_t - D_{xxx} \right) \tau_n \cdot \tau_{n-1} = 3\tau_{n-3}\tau_{n+2}. \tag{3.27}$$

Now let us compare our BTs with the standard ones which can be derived using the Hirota's bilinear approach as follows. Denoting the left-hand side of (2.4) by E,

$$E(\tau) = \left(D_{ty} - D_{xxxy} \right) \tau \cdot \tau \tag{3.28}$$

one can decompose the difference $E(\tau)\,\widetilde{\tau}^2 - \tau^2 E(\widetilde{\tau})$ as

$$E(\tau)\,\widetilde{\tau}^2 - \tau^2 E(\widetilde{\tau}) = 2D_y \; \left[\left(D_t - D_{xxx} \right) \tau \cdot \widetilde{\tau} \right] \cdot \tau\widetilde{\tau} + 6D_x \; \left(D_{xy}\, \tau \cdot \widetilde{\tau} \right) \cdot \left(D_x\, \tau \cdot \widetilde{\tau} \right). \tag{3.29}$$

This implies that, if functions τ and $\widetilde{\tau}$ are related by

$$\begin{cases} \left(D_t - D_{xxx} - 3\alpha \right) \tau \cdot \widetilde{\tau} = 0 & \alpha_y = 0 \\ \left(D_{xy} - \beta D_x \right) \tau \cdot \widetilde{\tau} = 0 & \beta_x = 0 \end{cases} \tag{3.30}$$

and $E(\tau) = 0$, then $E(\widetilde{\tau}) = 0$ as well, i.e. that system (3.30) describes a BT.

It was noted in the introduction that the BTs of the type discussed in this paper differ essentially from the usual BTs (3.30). This fact manifests itself in many aspects. Usually applications of BTs (3.30) lead from simple solutions to more complicated ones. The most bright example is the following. Let us apply (3.30) to the trivial tau-function $\tau = 1$. By solving the corresponding linear equations we come to the tau-function $\widetilde{\tau}$ which is a generalization of the one-soliton tau-function that can be obtained, say, by the Hirota's method. At the same time the simplest (vacuum) solution of the Volterra equations remains trivial (in the sense that it gives the zero solution of the NVVE) for all the values of n. The same can be said about more complex situations. BTs (3.30) at every step change the structure of solutions (add a soliton), while a shift of the Volterra index, $n \to n \pm 1$, usually leads to less essential changes. The only known exception is the determiant (Hankel, Wroskian etc) solutions where the index n is related to the size of the matrix. However these solutions have not been extended yet to satisfy the negative equations of the VH, so I cannot discuss them in the context of the NVVE.

The relationships between the BTs of both the kinds have been studied for the (1+1)-dimensional integrable systems such as the modified KdV or nonlinear Schrodinger equations (see, e.g, book by Newell [10]). In the most transparent way the difference between them can be exposed in the framework of the inverse scattering approach. The soliton-adding BTs similar to (3.30) are related to the Darboux transformations of the associated linear problem: they add zeroes to the corresponding scattering data. That is why they are often called Backlund-Darboux transformations. Transformations similar to those proposed in this paper, which are described by integrable chains (for example, in the case of the nonlinear Schrodinger equation it is the Toda chain), are of different kind. They are related to the Schlesinger transformations of the scattering problem and change the asymptotics (monodromy) of the fundamental solutions as functions of the spectral parameter (see, e.g., chapter 5 of [10]). However the inverse scattering approach for the NVVE is out of the scope of this paper and more elaborated discussion of this question needs further studies.

However, in some cases the relationships between both kinds of transformations are rather transparent. One can show using equations (3.23) and (3.24) that Volterra tau-functions solve (in the case of the proper boundary conditions)

$$D_{xy}\,\tau_n \cdot \tau_{n-1} + D_y\,\tau_{n+1} \cdot \tau_{n-2} = 0. \tag{3.31}$$

Taking this equation together with the third Volterra equation (3.27) one can show that Volterra tau-functions satisfy the system

$$\left\{ \begin{array}{l} (D_t - D_{xxx})\,\tau_n \cdot \tau_{n-1} - 3\tau_{n-3}\tau_{n+2} = 0 \\ D_{xy}\,\tau_n \cdot \tau_{n-1} + D_y\,\tau_{n+1} \cdot \tau_{n-2} = 0 \end{array} \right. \tag{3.32}$$

which can be reduced to the one similar to (3.30). Indeed, in the three-periodic case

$$\tau_{n+3} = \tau_n \tag{3.33}$$

equations (3.32) become

$$\left\{ \begin{array}{l} (D_t - D_{xxx} - 3)\,\tau_n \cdot \tau_{n-1} = 0 \\ D_{xy}\,\tau_n \cdot \tau_{n-1} = 0. \end{array} \right. \tag{3.34}$$

(note that (i) the VH $\rightarrow$ NVVE correspondence is valid for *any* solutions of the VH and that (ii) periodic reductions are compatible with the Volterra equations). Comparing (3.34) with (3.30) one can easily note that (3.34) coincide with (3.30) with $\alpha = 1$ and $\beta = 0$. So, one can conclude that the BTs described by the three-periodic Volterra equations are a particular case of the classical BTs (3.30). This situation, when Schlesinger transformations turn out to be some special (sometimes singular) cases of the Backlund-Darboux transformations, has been already discussed in the literature (see, e.g., [11, 12]). Of course the calculations presented above cannot give an exhaustive analysis of the relations between these BTs. I repeat, they are restricted to the three-periodic case, when the Volterra chain, I would like to recall, is nothing but a Painleve, namely the PIV, equation which is known to posses a number of peculiar features. Thus, the question of how to construct the BTs described by the VH of the Backlund-Darboux transformations remains to be settled and surely deserves further studies (probably in a more general framework).

4 From VH to NNVE.

The way from the VH to the NNVE is more straightforward than the calculations described in the previous sections: one has to write down three equations of the extended VH and show that any of their common solution also satisfies equations (1.1) or (2.4).

The VH is an infinite set of differential-difference equations compatible with the discrete problem

$$\psi_{n-1} - \psi_n + \zeta u_n \psi_{n+1} = 0, \qquad u_n = \frac{\tau_{n-2}\tau_{n+1}}{\tau_{n-1}\tau_n}. \tag{4.1}$$

The classical (or 'positive') Volterra equations are evolution equations $\partial u_n / \partial t_j = F_n^{(j)}$, two of which have already been written down (see (3.26) and (3.27)). Another type of the Volterra eqautions, which are non-local and which when taken together form the 'negative' VH, were discussed in [9]. The simplest of them is (3.24). The very important point is the fact that all equations of the extended VH (both the classical and the 'negative' ones) *are compatible.* So,

we can consider them simultaneously as one infinite system. We can think of u_n (or τ_n) as functions of an infinite number of times, $\tau_n = \tau_n(t_1, t_2, ..., \bar{t}_1, ...)$, where the dependence on t_j ($\bar{t}_j$) is determined by the jth 'positive' ('negative') Volterra equation.

Hereafter I will restrict myself to the finite subsystem of the VH, consisting of the equations mentioned above: the first and the third equations of the 'positive' subhierarchy (the corresponding times will be denoted by x and t) and the first 'negative' equation (with y used instead of $\bar{t}_1$). So, we will deal with the system

$$\begin{aligned} D_x\,\tau_n \cdot \tau_{n-1} &= \tau_{n-2}\tau_{n+1} && (4.2)\\ (D_t - D_{xxx})\,\tau_n \cdot \tau_{n-1} &= 3\tau_{n-3}\tau_{n+2} && (4.3)\\ D_y\,\tau_{n+1} \cdot \tau_{n-1} &= \tau_n^2 && (4.4) \end{aligned}$$

and before proceeding further I would like to return to the question of the compatibility of system (2.6)–(2.8). It turns out that we do not need to prove this fact separately: system (2.6)–(2.8) is equivalent to (4.2)–(4.4) which is a part of the VH, while consistency of the later has already been established (compatibility of equations of an integrable hierarchy, or commutativity of the corresponding flows, is one of the ingredients of its integrability).

The main result of this section, the transition from the VH to NNVE, can be achieved by the following simple calculations. Returning from the Hirota's bilinear differential operators to usual ones, one can present equation (4.3) with the help of (4.2) as

$$(\partial_t - \partial_{xxx}) \ln \frac{\tau_n}{\tau_{n-1}} = 3u_{n+1}u_n u_{n-1} + 3u_{n+1}u_n^2 + 3u_n^2 u_{n-1} + u_n^3 \qquad (4.5)$$

which gives for the quantity p_n,

$$p_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2} \qquad (4.6)$$

the identity

$$(\partial_t - \partial_{xxx})\, p_n = 6\,\partial_x (w_n p_n) \qquad (4.7)$$

where

$$w_n = \frac{\tau_{n-2}\tau_{n+2}}{\tau_n^2} = u_n u_{n+1}. \qquad (4.8)$$

On the other hand, (4.4) leads to

$$\partial_y w_n = p_n (u_n - u_{n+1}) \qquad (4.9)$$

Applying (4.2) again one can easily show that the right-hand side of the last equation is nothing but $-\partial_x p_n$. So,

$$\partial_y w_n + \partial_x p_n = 0. \qquad (4.10)$$

Comparing (4.7) and (4.10) with (1.1) one can see that *for any* n functions p_n and w_n solve the NNVE.

Using the fact that $w_n = \partial_{xx} \ln \tau_n$ and $p_n = -\partial_{xy} \ln \tau_n$ (these formulae can be derived from (4.2) and (4.4) after neglecting some unessential constants, which can be incorporated in the definition of τ_n) we can reformulate this result as follows: for each n the tau-function of the VH, τ_n, is a solution of the bilinear NNVE (2.4). In the context of the NNVE the meaning of the Volterra index n is clear: n is the number of our solution in the sequence of the BTs discussed in the previous section.

5 BTs and zero-curvature representation.

So far we have discussed the BTs in terms of solutions of our nonlinear equation only: both (2.6)–(2.8) and (3.30) are expressions which establish some links between different solutions of the NNVE. However, the structure of BTs for nonlinear integrable systems becomes more transparent when expressed in terms of the solutions of auxiliary linear problems. For example the NNVE can be presented as the compatibility condition for the system

$$\left\{ \begin{array}{lcl} \varphi_{xy} & = & 2p\varphi \\ \varphi_t & = & \varphi_{xxx} + 6w\varphi_x \end{array} \right. \tag{5.1}$$

(the so-called zero-curvature representation) and transfom (3.30) $\tau \to \widetilde{\tau}$ with $\beta = 0$,

$$D_{xy}\,\tau \cdot \widetilde{\tau} = 0 \tag{5.2}$$

leads to the following transformation of φ

$$\varphi \to \widetilde{\varphi}: \qquad \left\{ \begin{array}{l} \widetilde{\varphi}_x - \Lambda_x \widetilde{\varphi} = -\varphi_x - \Lambda_x \varphi \\ \widetilde{\varphi}_y - \Lambda_y \widetilde{\varphi} = \quad \varphi_y + \Lambda_y \varphi \end{array} \right. \qquad \Lambda = \ln \frac{\tau}{\widetilde{\tau}} \tag{5.3}$$

(these transformations are known as Loewner transformations [13] and were discussed e.g. in [14, 15]). I cannot present simple formulae for the Volterra sequence of the BTs in the framework of auxiliary problem (5.1). It turns out that to describe the BTs of this paper it is more convenient to use another linear problem associated with the NNVE which can be derived from the zero-curvature representation of the VH.

It can be shown that evolution of the functions ψ_n from (4.1) with respect to the flows described by (4.2)–(4.4) can be written as

$$\partial_x \psi_n = u_n \left(\psi_{n+1} - \psi_n \right) \tag{5.4}$$

$$\partial_y \psi_n = \frac{1}{p_{n-1}} \left(\psi_{n-1} - \psi_n \right) \tag{5.5}$$

$$\partial_t \psi_n = u_n u_{n+1} u_{n+2} \psi_{n+3} + \alpha_n \psi_{n+2} + \beta_n \psi_{n+1} + \gamma_n \psi_n \tag{5.6}$$

where u_n and p_n are defined in (3.25) and (4.6), while α_n, β_n and γ_n are some coefficients not written here explicitly. Equations (5.4) and (5.5) lead to

$$\partial_{xy} \psi_n = p_n \left(-\psi_{n-1} + 2\psi_n - \psi_{n+1} \right) \tag{5.7}$$

which can be transformed to

$$\partial_{xy} \psi_n + \frac{1}{p_{n-1}} \partial_x \psi_n + u_n \partial_y \psi_n = 0. \tag{5.8}$$

At the same time equation (5.6) can be presented as

$$\partial_t \psi_n = \partial_{xxx} \psi_n + 3 \left(u_{n-1} + u_n \right) \partial_{xx} \psi_n + 3 \left[\left(u_{n-1} + u_n \right)^2 + w_n \right] \partial_x \psi_n \tag{5.9}$$

where $w_n = u_n u_{n+1}$ (see (4.8)). Note that the last two equations can be rewritten in the 'one-site' form

$$\partial_{xy} \psi_n = \mu_n \partial_x \psi_n + \nu_n \partial_y \psi_n \tag{5.10}$$

$$\partial_t \psi_n = \partial_{xxx} \psi_n + 3a_n\, \partial_{xx} \psi_n + 3 \left(a_n^2 + w_n \right) \partial_x \psi_n \tag{5.11}$$

where

$$\mu_n = \frac{\partial_y h_n}{h_n}, \qquad \nu_n = \frac{\partial_{xy} h_n}{2\,\partial_y h_n}, \qquad a_n = -\frac{\partial_x h_n}{h_n} \tag{5.12}$$

with

$$h_n = \frac{\tau_{n-2}}{\tau_n}. \tag{5.13}$$

Equations (5.10) and (5.11) can be used as a zero-curvature representation of the NNVE, different from the traditional one given by (5.1). Indeed, one can check by straightforward calculatioins that the compatibility condition for the system

$$\psi_{xy} = \mu\psi_x + \nu\psi_y \tag{5.14}$$

$$\psi_t = \psi_{xxx} + 3a\psi_{xx} + 3\left(a^2 + w\right)\psi_x \tag{5.15}$$

where

$$\mu = \frac{h_y}{h}, \qquad \nu = \frac{h_{xy}}{2h_y}, \qquad a = -\frac{h_x}{h} \tag{5.16}$$

can be reduced to the system (1.1) for the quantities w and $p = \mu\nu$.

In terms of this zero-curvature representation it is clearly seen that the Volterra sequence of BTs discussed in this paper can be constructed by means of rising/lowering operators $\psi_n \to \psi_{n\pm1}$ given by (5.4) and (5.5):

$$\psi_{n+1} = \left(\frac{1}{u_n}\partial_x + 1\right)\psi_n \tag{5.17}$$

$$\psi_{n-1} = \left(p_{n-1}\partial_y + 1\right)\psi_n. \tag{5.18}$$

Thus our sequence of BTs for NNVE is generated by the iteration of the Laplace-Darboux transformations for linear problem (5.14) and (5.15).

6 Conclusion.

In this paper I have derived the BTs for the NNVE and have shown that these BTs can be iterated and that the resulting sequence can be described by the Volterra equations. The same result can be reformulated in a more general way: the NNVE can be embedded in the extended Volterra hierarchy by presenting it as a result of the combined action of the Volterra flows.

At the end I want to give the following remark. Among the three Volterra flows we used there were two 'positive' ones, $\partial/\partial t_1$ and $\partial/\partial t_3$. So, an interesting question is about the role of the 'skipped' second Volterra flow, $\partial/\partial t_2$. In terms of the NNVE, $\partial/\partial t_2$ can be viewed as some nonlocal symmetry. This symmetry could be used to derive the BTs, but I preferred more standard approach of introducing additional tau-functions (σ, ρ, ...) instead of introducing additional independent variables (t_2 and other). However, it should be noted that the second strategy is already known and had been shown, say, in [16] to be rather useful in a wide range of situations. The question of t_2-dependence is also interesting because it leads us to the Kadomtsev-Petviashvili (KP) equation. Indeed, it can be shown that any tau-function τ_n of the Volterra hierarchy solves also

$$\left(4D_1D_3 - 3D_2^2 - D_1^4\right)\,\tau_n \cdot \tau_n = 0 \tag{6.1}$$

where D_j are the Hirota operators corresponding to t_j, which means that τ_n is a tau function of the KP equation as well (or, in other words, that the KP equation can be embedded in the VH). So, the NNVE can be considered as a (nonlocal) symmetry of the KP equation. This fact can enlarge the area of application of the former and clarify its place among other integrable partial differential equations.

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Appendix A.

A proof of the fact that (2.6)–(2.8) are indeed the BTs can be given as follows. For the quantity E defined by (3.28),

$$E(\tau) = (D_{ty} - D_{xxxy})\ \tau \cdot \tau, \tag{A.2}$$

one can get, using explicit expressions for the Hirota's bilinear operators, that

$$\frac{E(\tau)}{6\tau^2} - \frac{E(\hat{\tau})}{6\hat{\tau}^2} = \frac{1}{3}\Lambda_{ty} - \frac{1}{3}\Lambda_{xxxy} - \Lambda_{xy}M_{xx} - \Lambda_{xx}M_{xy}. \tag{A.3}$$

Substitution of Λ_t from (2.6) gives

$$\frac{E(\tau)}{6\tau^2} - \frac{E(\hat{\tau})}{6\hat{\tau}^2} = \left(\frac{\lambda_{xx}\hat{\lambda}_{xx}}{\Lambda_x}\right)_y + \Lambda_x M_{xxy} - \Lambda_{xx}M_{xy} + \Lambda_x^2\Lambda_{xy}. \tag{A.4}$$

To proceed further we derive from (2.7) and (2.8) expressions for λ_{xxy} and $\hat{\lambda}_{xxy}$. After differentiating with respect to x equation (2.7) becomes

$$\lambda_{xxy}\hat{\lambda}_{xy} + \lambda_{xy}\hat{\lambda}_{xxy} - \Lambda_{xx} = 0. \tag{A.5}$$

Adding/subtracting (2.8) to/from this identity one can get

$$\begin{aligned} 0 &= \lambda_{xxy}\hat{\lambda}_{xy} + \Lambda_x^2 - \lambda_{xx} && \text{(A.6)}\\ 0 &= \lambda_{xy}\hat{\lambda}_{xxy} - \Lambda_x^2 + \hat{\lambda}_{xx} && \text{(A.7)}\end{aligned}$$

which leads, after multiplying (A.6) by λ_{xy}, (A.7) by $\hat{\lambda}_{xy}$ and using (2.7), to

$$\begin{aligned} \Lambda_x\lambda_{xxy} &= \lambda_{xy}\left(\lambda_{xx} - \Lambda_x^2\right) && \text{(A.8)}\\ \Lambda_x\hat{\lambda}_{xxy} &= \hat{\lambda}_{xy}\left(\Lambda_x^2 - \hat{\lambda}_{xx}\right). && \text{(A.9)}\end{aligned}$$

Now one can calculate $\left(\lambda_{xx}\hat{\lambda}_{xx}/\Lambda_x\right)_y$:

$$
\begin{aligned}
\left(\frac{\lambda_{xx}\hat{\lambda}_{xx}}{\Lambda_x}\right)_y &= \frac{\lambda_{xx}}{\Lambda_x}\hat{\lambda}_{xxy} + \frac{\hat{\lambda}_{xx}}{\Lambda_x}\lambda_{xxy} - \frac{\lambda_{xx}\hat{\lambda}_{xx}}{\Lambda_x^2}\Lambda_{xy} &\text{(A.10)}\\
&= \frac{\lambda_{xx}\hat{\lambda}_{xy}}{\Lambda_x^2}\left(\Lambda_x^2 - \hat{\lambda}_{xx}\right) + \frac{\lambda_{xy}\hat{\lambda}_{xx}}{\Lambda_x^2}\left(\lambda_{xx} - \Lambda_x^2\right) - \frac{\Lambda_{xy}}{\Lambda_x^2}\lambda_{xx}\hat{\lambda}_{xx} &\text{(A.11)}\\
&= \lambda_{xx}\hat{\lambda}_{xy} - \lambda_{xy}\hat{\lambda}_{xx}. &\text{(A.12)}
\end{aligned}
$$

On the other hand, summarizing (A.8) and (A.9) one can get

$$
\begin{aligned}
\Lambda_x M_{xxy} &= \lambda_{xx}\lambda_{xy} - \hat{\lambda}_{xx}\hat{\lambda}_{xy} - \Lambda_x^2\Lambda_{xy} &\text{(A.13)}\\
&= \lambda_{xx}\left(M_{xy} - \hat{\lambda}_{xy}\right) + \hat{\lambda}_{xx}\left(\lambda_{xy} - M_{xy}\right) - \Lambda_x^2\Lambda_{xy} &\text{(A.14)}
\end{aligned}
$$

which leads to

$$
\Lambda_x M_{xxy} - \Lambda_{xx}M_{xy} + \Lambda_x^2\Lambda_{xy} = \lambda_{xy}\hat{\lambda}_{xx} - \lambda_{xx}\hat{\lambda}_{xy}. \tag{A.15}
$$

Comparing (A.12) and (A.15) one can conclude that the right-hand side of (A.4) equals to zero. This means that

$$
E(\tau) = 0 \qquad \Rightarrow \qquad E(\hat{\tau}) = 0 \tag{A.16}
$$

if τ and $\hat{\tau}$ satisfy (2.6)–(2.8). This completes the proof of the fact that relations (2.6)–(2.8) are indeed a BT for the NNVE.

Appendix B

Here I discuss a proof of the fact that finite system (3.21), (3.22) can be extended to infinity. Consider first the bilinear quantities V_n, $\overline{V}_n$ and C_n defined by

$$
\begin{aligned}
V_n &= D_x\,\tau_n \cdot \tau_{n-1} - \tau_{n+1}\tau_{n-2} &\text{(B.1)}\\
\overline{V}_n &= D_y\,\tau_{n+1} \cdot \tau_{n-1} - \tau_n^2 &\text{(B.2)}\\
C_n &= \frac{1}{2}D_{xy}\,\tau_n \cdot \tau_n + \tau_{n+1}\tau_{n-1} &\text{(B.3)}
\end{aligned}
$$

From the definition of the Hirota's operators one can derive by simple algebra the fourth-order identities

$$
\tau_n\tau_{n-1}V_{n+1} - \tau_{n+1}\tau_n V_n - D_x\,\overline{V}_n \cdot \tau_{n+1}\tau_{n-1} + \tau_{n-1}^2 C_{n+1} - \tau_{n+1}^2 C_{n-1} = 0 \tag{B.4}
$$

and

$$
\tau_{n+1}\tau_{n-1}\overline{V}_{n+1} - \tau_{n+2}\tau_n\overline{V}_n + D_y\,V_{n+1} \cdot \tau_{n+1}\tau_n - \tau_n^2 C_{n+1} + \tau_{n+1}^2 C_n = 0 \tag{B.5}
$$

that are satisfied by *any* sequences of τ_n's. My aim now is to show that if $V_n = \overline{V}_n = 0$ for a sufficiently large number of sequential values of n, then all of the V_n and $\overline{V}_n$ are equal to zero as well. To do that I rewrite (B.4) and (B.5) as

$$
\begin{aligned}
&\frac{1}{p_n}\left(Y_{n+1} - Y_n\right) - \partial_x\overline{Y}_n + Z_{n+1} - Z_{n-1} = 0 &\text{(B.6)}\\
&u_n\left(\overline{Y}_{n+1} - \overline{Y}_n\right) + \partial_x Y_{n+1} - Z_{n+1} + Z_n = 0 &\text{(B.7)}
\end{aligned}
$$

where

$$Y_n = \frac{V_n}{\tau_{n-1}\tau_n}, \qquad \overline{Y}_n = \frac{\overline{V}_n}{\tau_{n-1}\tau_{n+1}}, \qquad Z_n = \frac{C_n}{\tau_n^2} \tag{B.8}$$

(I presume that none of τ_n is equal to zero) from which it follows that

$$\frac{1}{p_n}\left(Y_{n+1} - Y_n\right) + \partial_x\left(Y_{n+1} + Y_n\right) = u_{n+1}\left(\overline{Y}_n - \overline{Y}_{n+1}\right) + u_n\left(\overline{Y}_{n-1} - \overline{Y}_n\right) + \partial_x \overline{Y}_n \tag{B.9}$$

or

$$\overline{Y}_{n+1} = lin\left(Y_{n+1}, Y_n, \overline{Y}_n, \overline{Y}_{n-1}\right) \tag{B.10}$$

where $lin(...)$ is a linear combination of its arguments (and their derivatives).

Now we can return to the Volterra equations. The finite Volterra system (3.21) and (3.22) can be written as

$$Y_0 = Y_1 = Y_2 = 0 \qquad \text{and} \qquad \overline{Y}_{-1} = \overline{Y}_0 = \overline{Y}_1 = \overline{Y}_2 = 0 \tag{B.11}$$

So, if we *define* τ_4 as $\tau_4 = \left(D_x\,\tau_3 \cdot \tau_2\right)/\tau_1$ (which means that $Y_3 = 0$), then by virtue of (B.10) we get $\overline{Y}_3 = 0$. Repeating this procedure we can define an infinite set of τ_n in such a way that

$$Y_n = \overline{Y}_n = 0 \qquad \text{for} \qquad n \geq 0 \tag{B.12}$$

It is also possible to *define* tau-functions for $n \leq -3$ to ensure vanishing of all Y_n and $\overline{Y}_n$ for $n < 0$. This means that these tau-functions will be solutions of the first positive and first negative Volterra equations:

$$V_n = 0, \quad \overline{V}_n = 0 \qquad \text{for} \qquad n = 0, \pm 1, \pm 2, ... \tag{B.13}$$

In a similar way one can consider the chains of identities for the bilinear combinations of tau-functions which generate the t-equation (3.27) of the VH.

References

[1] Nizhnik L.P., 1980, Integration of multidimensional nonlinear equations by the method of inverse problem. *DAN SSSR*, **254**, 332-335 (in Russian). Translated in: *Sov.Phys.Dokl.*, **25**, 706-708.

[2] Veselov A.P. and Novikov S.P., 1984, Finite-gap two-dimensional potential Schrödinger operators. Explicit formulae and evolution equations. *DAN SSSR*, **279**, 20-24 (in Russian). Translated in: *Sov.Math.Dokl.*, **30**, 588-591.

[3] Dorfman I.Y. and Athorne C., 1993, On the nonsymmetric Novikov-Veselov hierarchy. *Phys. Letters A*, **182**, 369-372.

[4] Levi D. and Benguria D., 1980, Backlund transformations and nonlinear differential difference equations. *Proc. Natl. Ac. Sci.* **77**, 5025-5027.

[5] Levi D., 1981, Nonlinear differential-difference equations as Backlund transformations. *J. Phys. A: Math. Gen.*, **14**, 1083-1098.

[6] Flaschka H., 1983, Relations between infinite-dimensional and finite-dimensional isospectral equations. *Non-linear integrable systems - classical theory and quantum theory*, Proc. RIMS Symp., Kyoto 1981, 219-240.

[7] Shabat A.B. and Yamilov R.I., 1988, Lattice representations of integrable systems. *Phys. Lett.*, **130A**, 271-275.

[8] Jimbo M. and Miwa T., 1983, Solitons and infinite dimensional algebras. *Publ. RIMS*, **19**, 943-1001.

[9] Pritula G.M. and Vekslerchik V.E., 2003, Negative Volterra flows. *J. Phys. A: Math. Gen.*, **36**, 213-226.

[10] Newell A.C., 1985, *Solitons in Mathematics and Physics*, (Philadelphia: SIAM).

[11] Leble S.B., 2002, Elementary, binary and Schlesinger transformations in differential ring geometry. *European Phys. J. B*, **29**, 189-192.

[12] Adam Doliwa, Manuel Manas, Luis Martnez Alonso, Elena Medina and Paolo Maria Santini, 1999, Charged free fermions, vertex operators and the classical theory of conjugate nets. *J. Phys. A: Math. Gen.*, **32**, 1197-1216

[13] Loewner C., 1952, Generation of solutions of systems of partial differential equations by composition of infinitesimal Backlund transformations. *J. Anal. Math.*, **2**, 219-242.

[14] Konopelchenko B. and Rogers C., 1993, On generalized Loewner systems: novel integrable equations in 2+1-dimensions. *J. Math. Phys.*, **34**, 214-242.

[15] Schief W.K. and Rogers C., 1998, Loewner transformations: adjoint and binary Darboux connections. *Stud. Appl. Math.*, **100**, 391-422.

[16] Lambert F., Loris I., Springael J. and Willox R., 2001, On the Hirota representation of soliton equations with one tau-function. *J. Phys. Soc. Japan*, **70**, 605-608.